Perturbations in inflationary cosmologies with smooth exit

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Abstract

We give a comprehensive analysis of how scalar and tensor perturbations evolve in cosmologies with a smooth transition from power-law-like and de Sitter-like inflation to a radiation era. Analytic forms for the super-horizon and sub-horizon perturbations in the inflationary and radiation dominated eras are found.

I Introduction

In [1], the evolution of perturbations for a decaying vacuum cosmology given in [2], with smooth exit from inflation to radiation-domination, was found exactly on super-horizon scales. The solution confirmed the standard result [3, 4, 5, 6, 7] within gauge invariant perturbation theory that super-horizon density perturbations are strongly amplified through the transition from inflation to radiation. This transition is often approximated as an instantaneous jump [3, 4, 7], since super-horizon modes change on a timescale that is much greater than the transition time. To avoid problems in the matching conditions for a jump transition, we consider only smooth transitions.

Given the variety of methods and models that have been used, we believe it is useful to do a comparative study, using the same method on two broad classes of transition models, i.e. those based on power-law-like and de Sitter-like inflation. We use the class of power-law-like models first given in [8]. The second class of models is a generalisation of the models of [5, 8] with a de Sitter-like inflationary era. The models, together with the neccessary equations in perturbation theory, are discussed in section 2. In section 3 we present our results for scalar and tensor perturbations on super-horizon scales, and show how they confirm the standard predictions. In section 4 we present solutions for scalar and tensor perturbations on sub-horizon scales. However, toy models that evolve smoothly from inflation to radiation dominated eras are only applicable for super-horizon modes around the time of transition, since on sub-horizon scales, the dynamics of the reheating era have a significant effect. The solutions for sub-horizon scales therefore do not apply during the transition era. Section 5 gives some concluding remarks.

Throughout this paper we use units in which $8\pi G = 1 = c$.

II The models and perturbation equations

A flat Friedmann-Lemaitre-Robertson-Walker (FLRW) background has metric

$$ds^2 = -dt^2 + a^2(t)[dx^2 + dy^2 + dz^2], \tag{1}$$

where t is the physical time and $a(t)$ is the expansion scale factor. The background pressure p, energy density ρ and Hubble parameter $H = \dot{a}/a$ (where a dot denotes a derivative with respect to cosmic time t) satisfy the field equations

$$p = -2\dot{H} - 3H^2, \tag{2}$$
$$\rho = 3H^2. \tag{3}$$

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The pressure index and effective adiabatic sound speed are defined by

$$w = \frac{p}{\rho}, \tag{4}$$

$$c_{\mathrm{s}}^2 = \frac{dp}{d\rho} = \frac{\dot{p}}{\dot{\rho}} \tag{5}$$

Inflation is characterised by $\ddot{a} > 0$, so that the epoch t_{e} of exit is given by $\ddot{a}(t_{\mathrm{e}}) = 0$. The radiation dominated era is characterised by $p \approx \frac{1}{3}\rho$, $H \approx 1/2t$.

Scalar and tensor perturbations lead to a metric of the form

$$ds^2 = a^2(\eta)\{-(1 + 2\Phi)d\eta^2 + [(1 - 2\Psi)\gamma_{ij} + h_{ij}]dx^i dx^j\}, \tag{6}$$

in the longitudinal gauge, where η is conformal time ($d\eta = dt/a$) [4]. Here Φ and Ψ are gauge-invariant amplitudes of the scalar metric perturbations. The field equations require $\Phi = \Psi$ when the spatial part of the energy-momentum tensor is diagonal. The tensor perturbations are gauge-invariantly described by $h_{ij} = hQ_{ij}$, where Q_{ij} is a symmetric traceless three tensor and h represents the amplitude of a gravitational wave.

The evolution of adiabatic scalar perturbations Φ is given by [4]

$$\frac{d^2\Phi}{d\eta^2} + 3aH(1 + c_{\mathrm{s}}^2)\frac{d\Phi}{d\eta} - \nabla^2\Phi + 3a^2H^2(c_s^2 - w)\Phi = 0. \tag{7}$$

Decomposing $\Phi(\eta, \vec{x})$ into eigenmodes $\tilde{\Phi}(\eta, \vec{k})$ of the comoving Laplacian, and then using the cosmic time t and scale factor a as the dynamical variables respectively, gives

$$\frac{d^2\tilde{\Phi}}{dt^2} + 3H\left(\frac{4}{3} + c_{\mathrm{s}}^2\right)\frac{d\tilde{\Phi}}{dt} + \left[3H^2(c_s^2 - w) + \left(\frac{k}{a}\right)^2\right]\tilde{\Phi} = 0, \tag{8}$$

$$\frac{d^2\tilde{\Phi}}{da^2} + \frac{1}{a}\left(\frac{7}{2} + 3c_s^2 - \frac{3}{2}w\right)\frac{d\tilde{\Phi}}{da} + \left[\frac{3}{a^2}(c_s^2 - w) + \frac{k^2}{a^4H^2}\right]\tilde{\Phi} = 0, \tag{9}$$

where k is the comoving wave number. The decomposition allows us to follow the evolution of a single mode through the inflationary and radiation dominated eras. The physical length scale λ correponding to the comoving wavenumber k is $\lambda = 2\pi a/k$. This will equal the Hubble radius H^{-1} when

$$k = 2\pi aH. \tag{10}$$

Super-horizon modes are characterised by $k \ll aH$. For these modes one can neglect the k^2 term in equations (8) and (9) (and subsequent equations). The solutions of the truncated equations will be a good approximation to the limit of the solutions of the full equations, provided [9]

$$k^2\Phi \to 0 \text{ as } k \to 0. \tag{11}$$

For adiabatic super-horizon scalar perturbations, the growing modes have a conserved quantity [10] which can be used to determine $\tilde{\Phi}$:

$$\zeta \equiv \tilde{\Phi} + \frac{2}{3(1 + w)}\left(\tilde{\Phi} + \frac{1}{H}\frac{d\tilde{\Phi}}{dt}\right) \Rightarrow \tilde{\Phi} = -\zeta\frac{H}{a}\int\frac{a\dot{H}}{H^2}dt. \tag{12}$$

Equation (12) can be used to express the growing perturbations at late times in terms of their early time forms [4]. In [5, 6, 7], ζ is used to estimate the amplification of perturbations through a smooth transition. This amplification has been disputed in [11] (see also [12, 13]).

The density perturbations in the longitudinal gauge are given by [4]:

$$\frac{\delta\rho}{\rho} = -2\left(\frac{k^2}{3a^2H^2} + 1\right)\tilde{\Phi} - \frac{2}{H}\frac{d\tilde{\Phi}}{dt}. \tag{13}$$

The Fourier mode $\tilde{h}$ of the amplitude h of the tensor perturbations satisfies the equation [4]

$$\frac{d^2\tilde{h}}{dt^2} + 3H\frac{d\tilde{h}}{dt} + \left(\frac{k}{a}\right)^2 \tilde{h} = 0\,, \tag{14}$$

which may also be re-written with a as the dynamical variable:

$$a^2\frac{d^2\tilde{h}}{da^2} + \left(4a + \frac{a^2}{H}\frac{dH}{da}\right)\frac{d\tilde{h}}{da} + \frac{k^2}{a^4H^2}\tilde{h} = 0\,. \tag{15}$$

For super-horizon modes one can reduce to quadrature:

$$\tilde{h} = C + D\int\frac{dt}{a^3} = C + D\int\frac{da}{a^4H}\,, \tag{16}$$

where C and D are constant. It follows that D corresponds to the decaying mode and therefore there is no amplification in long-wavelength tensor perturbations [3, 4].

A Power-law-like models

In [8] a simple class of power-law-like inflation models, which exit smoothly to a radiation era, is given by

$$a(t) = \frac{a_{\mathrm{e}}}{\sqrt{t_{\mathrm{e}}}}(1+m)^{(2n-1)/2}t^n(t+mt_{\mathrm{e}})^{(1-2n)/2}\,, \tag{17}$$

where the parameter n satisfies $n > 1$ to acheive inflationary expansion, and

$$m = \frac{n + \sqrt{n(2n-1)}}{2n(n-1)}\,.$$

For $t \ll mt_{\mathrm{e}}$,

$$a \approx a_{\mathrm{e}}\left(1 + \frac{1}{m}\right)^{(2n-1)/2}\left(\frac{t}{t_{\mathrm{e}}}\right)^n\,,$$

showing the early power-law-like inflation era. For $t \gg mt_{\mathrm{e}}$,

$$a \approx a_{\mathrm{e}}(1+m)^{(2n-1)/2}\left(\frac{t}{t_{\mathrm{e}}}\right)^{1/2}\,,$$

as in a radiation dominated era. Thus we have a cosmology which evolves smoothly from power-law-like inflation to a radiation dominated era. The Hubble rate follows on differentiating equation (17):

$$H(t) = \frac{t + 2nmt_{\mathrm{e}}}{2t(t+mt_{\mathrm{e}})}\,. \tag{18}$$

Using equations (2) and (3), equation (18) gives

$$p = \frac{t^2 + 4nmt_{\mathrm{e}}t - 4n(3n-2)m^2t_{\mathrm{e}}^2}{4t^2(t+mt_{\mathrm{e}})^2}\,, \tag{19}$$

$$\rho = \frac{3(t+2nmt_{\mathrm{e}})^2}{4t^2(t+mt_{\mathrm{e}})^2}\,. \tag{20}$$

It follows that the pressure index and effective adiabatic sound speed given by equations (4) and (5) are

$$w = \frac{t^2 + 4nmt_{\mathrm{e}}t - 4n(3n-2)m^2t_{\mathrm{e}}^2}{3t^2 + 12nmt_{\mathrm{e}}t + 12n^2m^2t_{\mathrm{e}}^2}\,, \tag{21}$$

$$c_{\mathrm{s}}^2 = \frac{t(t+mt_{\mathrm{e}})(t+2nmt_{\mathrm{e}}) - [t^2 + 4nmt_{\mathrm{e}}t - 4n(3n-2)m^2t_{\mathrm{e}}^2](2t+mt_{\mathrm{e}})}{3t(t+mt_{\mathrm{e}})(t+2nmt_{\mathrm{e}}) - 3(t+2nmt_{\mathrm{e}})^2(2t+mt_{\mathrm{e}})}\,. \tag{22}$$

The evolution of scalar and tensor perturbations for this model is given by equations (8), (13) and (14). Using equations (17), (18), (21) and (22), we get

$$t^2 \frac{d^2\tilde{\Phi}}{dt^2} + t \left[\frac{t + 2nmt_{\mathrm{e}}}{2(t + mt_{\mathrm{e}})}\right] U(t) \frac{d\tilde{\Phi}}{dt} + \left[\frac{t + 2nmt_{\mathrm{e}}}{2(t + mt_{\mathrm{e}})}\right]^2 V(t)\tilde{\Phi} = 0 \,, \tag{23}$$

where

$$
\begin{aligned}
U(t) &= \frac{5t(t + mt_{\mathrm{e}})(t + 2nmt_{\mathrm{e}}) - (5t^2 + 20nmt_{\mathrm{e}}t + 4n(n + 2)m^2 t_{\mathrm{e}}^2)(2t + mt_{\mathrm{e}})}{t(t + mt_{\mathrm{e}})(t + 2nmt_{\mathrm{e}}) - (t + 2nmt_{\mathrm{e}})^2(2t + mt_{\mathrm{e}})} \,, \\
V(t) &= \frac{t(t + mt_{\mathrm{e}})(t + 2nmt_{\mathrm{e}}) - [t^2 + 4nmt_{\mathrm{e}}t - 4n(3n - 2)m^2 t_{\mathrm{e}}^2](2t + mt_{\mathrm{e}})}{t(t + mt_{\mathrm{e}})(t + 2nmt_{\mathrm{e}}) - (t + 2nmt_{\mathrm{e}})^2(2t + mt_{\mathrm{e}})} \\
&\quad - \frac{t^2 + 4nmt_{\mathrm{e}}t - 4n(3n - 2)m^2 t_{\mathrm{e}}^2}{(t + 2nmt_{\mathrm{e}})^2} + k^2 \frac{t_{\mathrm{e}}(1 + m)^{1-2n} t^{2-2n}(t + mt_{\mathrm{e}})^{2n+1}}{a_{\mathrm{e}}^2(t + 2nmt_{\mathrm{e}})^2} \,,
\end{aligned}
$$

and

$$\frac{\delta\rho}{\rho} = -2 \left[\frac{4k^2 t_{\mathrm{e}}(1 + m)^{1-2n} t^{2-2n}(t + mt_{\mathrm{e}})^{2n+1}}{3a_{\mathrm{e}}^2(t + 2nmt_{\mathrm{e}})^2} + 1\right] \tilde{\Phi} - \left[\frac{4t(t + mt_{\mathrm{e}})}{t + 2nmt_{\mathrm{e}}}\right] \frac{d\tilde{\Phi}}{dt} \,, \tag{24}$$

and

$$
\begin{aligned}
& t^2 \frac{d^2\tilde{h}}{dt^2} + 3t \left[\frac{t + 2nmt_{\mathrm{e}}}{2(t + mt_{\mathrm{e}})}\right] \frac{d\tilde{h}}{dt} \\
& \quad + \left[\frac{k^2 t_{\mathrm{e}}(1 + m)^{1-2n}(t + mt_{\mathrm{e}})^{2n-1} t^{2-2n}}{a_{\mathrm{e}}^2}\right] \tilde{h} = 0 \,.
\end{aligned}
\tag{25}
$$

B   de Sitter-like models

In [8] a new exact inflationary solution is given, using a as the effective dynamical variable. The Hubble rate is given by

$$H(a) = \frac{2H_{\mathrm{e}}a_{\mathrm{e}}^2}{a^2 + a_{\mathrm{e}}^2} \,, \tag{26}$$

where H_{e} is the Hubble rate at exit. A similar model [but one which does not produce an exact solution for super-horizon scalar perturbations, unlike (26) [1]) is proposed in [5]:

$$H(a) = \frac{2^{1/2}H_{\mathrm{e}}a_{\mathrm{e}}^2}{(a^4 + a_{\mathrm{e}}^4)^{1/2}} \,. \tag{27}$$

For both Hubble rates, when $a \ll a_{\mathrm{e}}$, H is approximately constant as in a de Sitter inflation era, and for $a \gg a_{\mathrm{e}}$, the Hubble rate decays like a^{-2}, as in a radiation dominated era. Thus for both models we have a cosmology which evolves smoothly from inflation to a radiation dominated era.

Here we generalise these models by considering the class of Hubble rates

$$H(a) = \frac{2^{1/\ell}a_{\mathrm{e}}^2 H_{\mathrm{e}}}{(a^{2\ell} + a_{\mathrm{e}}^{2\ell})^{1/\ell}} \,, \tag{28}$$

parametrised by ℓ ($\ell > 0$). For equation (26), $\ell = 1$, and for (27), $\ell = 2$. Using equation (28), $H \approx$ constant for $a \ll a_{\mathrm{e}}$ (inflation), and $H \propto a^{-2}$ for $a \gg a_{\mathrm{e}}$ (radiation domination), confirming that the model has the desired properties for all positive ℓ.

Using equations (2) and (3), equation (28) gives

$$
\begin{aligned}
p &= \frac{2^{2/\ell}H_{\mathrm{e}}^2 a_{\mathrm{e}}^4(a^{2\ell} - 3a_{\mathrm{e}}^{2\ell})}{(a^{2\ell} + a_{\mathrm{e}}^{2\ell})^{1+2/\ell}} \,, \tag{29} \\
\rho &= \frac{2^{2/\ell}3H_{\mathrm{e}}^2 a_{\mathrm{e}}^4}{(a^{2\ell} + a_{\mathrm{e}}^{2\ell})^{2/\ell}} \,. \tag{30}
\end{aligned}
$$

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The pressure index and effective adiabatic sound speed given by equations (4) and (5) are

$$w = \frac{1}{3}\left(\frac{a^{2\ell} - 3a_{\mathrm{e}}^{2\ell}}{a^{2\ell} + a_{\mathrm{e}}^{2\ell}}\right), \tag{31}$$

$$c_{\mathrm{s}}^2 = \frac{1}{3}\left[\frac{a^{2\ell} - (2\ell + 3)a_{\mathrm{e}}^{2\ell}}{a^{2\ell} + a_{\mathrm{e}}^{2\ell}}\right]. \tag{32}$$

Applying equation (10) to equation (28) and taking $k_{\mathrm{e}} = 2\pi a_{\mathrm{e}}H_{\mathrm{e}}$ (the comoving wavenumber of the Hubble radius at exit), the epoch a_- of leaving and a_+ of re-entering the Hubble radius are given exactly in this class of models by

$$\frac{a_{\pm}}{a_{\mathrm{e}}} = \frac{k_{\mathrm{e}}}{k}\left[1 \pm \sqrt{1 - \left(\frac{k}{k_{\mathrm{e}}}\right)^{2\ell}}\right]^{1/\ell}. \tag{33}$$

Scales with $k \geq k_{\mathrm{e}}$ never cross the Hubble radius and remain sub-horizon. For super-horizon scales with $k \ll k_{\mathrm{e}}$, equation (33) reduces to

$$\frac{a_{\pm}}{a_{\mathrm{e}}} \approx \left(\frac{2^{1/\ell}k_{\mathrm{e}}}{k}\right)^{\pm 1}.$$

The scalar and tensor perturbations for this model are governed by equations (9), (13) and (15). Using equations (28), (31) and (32) we get

$$a^2\frac{d^2\tilde{\Phi}}{da^2} + a\left[\frac{4a^{2\ell} - (2\ell - 2)a_{\mathrm{e}}^{2\ell}}{a^{2\ell} + a_{\mathrm{e}}^{2\ell}}\right]\frac{d\tilde{\Phi}}{da}$$
$$+ \left[\pi^2\left(\frac{k}{k_{\mathrm{e}}}\right)^2\frac{(a^{2\ell} + a_{\mathrm{e}}^{2\ell})^{2/\ell}}{2^{2/\ell - 2}a_{\mathrm{e}}^2 a^2} - \frac{2\ell a_{\mathrm{e}}^{2\ell}}{(a^{2\ell} + a_{\mathrm{e}}^{2\ell})}\right]\tilde{\Phi} = 0, \tag{34}$$

and

$$\frac{\delta\rho}{\rho} = -2\left[\frac{1}{3}\left(\frac{k}{k_{\mathrm{e}}}\right)^2\frac{(a^{2\ell} + a_{\mathrm{e}}^{2\ell})^{2/\ell}}{2^{2/\ell}a_{\mathrm{e}}^2 a^2} + 1\right]\tilde{\Phi} - 2a\frac{d\tilde{\Phi}}{da}, \tag{35}$$

and

$$a^2\frac{d^2\tilde{h}}{da^2} + 2a\left[\frac{a^{2\ell} + 2a_{\mathrm{e}}^{2\ell}}{a^{2\ell} + a_{\mathrm{e}}^{2\ell}}\right]\frac{d\tilde{h}}{da} + \left[\pi^2\left(\frac{k}{k_{\mathrm{e}}}\right)^2\frac{(a^{2\ell} + a_{\mathrm{e}}^{2\ell})^{2/\ell}}{2^{2/\ell - 2}a_{\mathrm{e}}^2 a^2}\right]\tilde{h} = 0. \tag{36}$$

III Super-horizon perturbations

For the modes which leave the Hubble radius (during inflation), while they remain outside the Hubble radius, we can neglect the k-term in equations (23), (24), (25), (34), (35) and (36). Only equations (25) and (36), using (16), give exact solutions for the truncated equations. However, using [14, 15] we can give analytic forms for the solutions in the inflationary and radiation dominated eras for equations (23), (24), (34) and (35), since they reduce to Bessel forms. For all solutions, A_j, B_j, $j = 1, 2, ...$ are used to denote arbitrary constants. For equation (11) to be satisfied it is required that for scalar perturbations A_j, $B_j \sim k^q$ as $k \to 0$, where $q > -2$. For all scalar perturbation solutions this is assumed to be the case.

A Power-law-like model

For $t \ll mt_{\mathrm{e}}$ equation (23) has solution

$$\tilde{\Phi} \approx A_1 + B_1\left(\frac{t_{\mathrm{e}}}{t}\right)^{n+1}, \tag{37}$$

describing modes that leave well before exit, while for $t \gg mt_{\mathrm{e}}$

$$\tilde{\Phi} \approx A_2 + B_2\left(\frac{t_{\mathrm{e}}}{t}\right)^{3/2}, \tag{38}$$

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describing modes that remain super-horizon in the radiation era. In equations (37) and (38), A_j correspond to the constant modes, and B_j to the decaying modes.

The conserved quantity given by equation (12), applied to the case of the model given by (17), leads to

$$\tilde{\Phi} = \zeta \frac{(t + mt_{\mathrm{e}})^{(2n-3)/2}(t + 2nmt_{\mathrm{e}})}{t^{n+1}} \int \frac{t^n(t^2 + 4nmt_{\mathrm{e}}t + 2nm^2t_{\mathrm{e}}{}^2)}{(t + 2nmt_{\mathrm{e}})^2(t + mt_{\mathrm{e}})^{(2n-1)/2}} dt \,, \tag{39}$$

where we neglect the decaying contributions (and therefore no constant of integration arises from the integral). Amplification of the scalar growing modes according to equations (37) and (38) is given by the factor $\alpha = A_2/A_1$. Using equation (39), we find that

$$A_1 = \frac{\zeta}{(n+1)} \,,$$

and

$$A_2 = \frac{2}{3}\zeta \,.$$

Therefore

$$\alpha = \frac{2}{3}(n+1) > \frac{4}{3} \,, \tag{40}$$

so that amplification is only significant for large n. This is in agreement with results based on instantaneous transition models [3, 4, 7].

The density perturbations in the inflationary and radiation dominated eras, are found analytically, using equations (24), (37) and (38). For modes that leave the Hubble radius well before exit ($t \ll mt_{\mathrm{e}}$)

$$\frac{\delta\rho}{\rho} \approx -2A_1 + \frac{2}{n}B_1\left(\frac{t_{\mathrm{e}}}{t}\right)^{n+1} \,, \tag{41}$$

while for modes that are still super-horizon during radiation domination ($t \gg mt_{\mathrm{e}}$)

$$\frac{\delta\rho}{\rho} \approx -2A_2 + 4B_2\left(\frac{t_{\mathrm{e}}}{t}\right)^{3/2} \,. \tag{42}$$

The density perturbations undergo the same amplification (40) as the potential perturbations.

The tensor perturbations are given by equation (16):

$$\tilde{h} = A_3 + B_3 \int t^{-3n}(t + mt_{\mathrm{e}})^{3(2n-1)/2} dt \,. \tag{43}$$

During power-law inflation ($t \ll mt_{\mathrm{e}}$)

$$\tilde{h} \approx A_3 + B_3 \frac{m^{(6n-3)/2}t_e^{-1/2}}{1 - 3n}\left(\frac{t_{\mathrm{e}}}{t}\right)^{3n-1} \,, \tag{44}$$

while for those modes that remain super-horizon during radiation domination ($t \gg mt_{\mathrm{e}}$)

$$\tilde{h} \approx A_3 - 2B_3 t^{-1/2} \,. \tag{45}$$

The results (43)-(45) imply that tensor perturbations do not grow on super-horizon scales.

B de Sitter-like model

For modes that leave the Hubble radius when $a \ll a_{\mathrm{e}}$, equation (34) has super-horizon solutions

$$\tilde{\Phi} \approx A_4\left(\frac{a}{a_{\mathrm{e}}}\right)^{2\ell} + B_4\frac{a_{\mathrm{e}}}{a} \,. \tag{46}$$

For super-horizon modes during radiation domination $(a \gg a_\mathrm{e})$

$$\tilde{\Phi} \approx \left(\frac{a_\mathrm{e}}{a}\right)^{3/2} Z_{-3/2\ell}\left[-i\sqrt{\frac{2}{\ell}}\left(\frac{a_\mathrm{e}}{a}\right)^{\ell}\right] , \qquad (47)$$

where Z_ν denotes a linear combination of the Bessel functions J_ν and Y_ν. Using the asymptotic forms of the Bessel functions [14, 15], equation (47) reduces to

$$\tilde{\Phi} \approx A_5 + B_5\left(\frac{a_\mathrm{e}}{a}\right)^3 . \qquad (48)$$

It follows that in the inflationary era, $|\tilde{\Phi}|$ grows as $a^{2\ell}$, while in the radiation dominated era, $|\tilde{\Phi}|$ is approximately constant (while the scales are still super-horizon). Thus growing super-horizon scalar perturbations are strongly amplified during inflation and then remain approximately constant after inflation. The amplification factor is clearly much greater than that for the power-law-like model.

We can calculate the amplification using equation (12), which gives

$$\tilde{\Phi} = \frac{2\zeta}{a(a^{2\ell} + a_\mathrm{e}^{2\ell})^{1/\ell}} \int \frac{a^{2\ell}(a^{2\ell} + a_\mathrm{e}^{2\ell})^{1/\ell}}{(a^{2\ell} + a_\mathrm{e}^{2\ell})} da . \qquad (49)$$

Evaluating the integral in the inflationary and radiation era's and using equations (46) and (48), we find

$$A_4 = \frac{2\zeta}{2\ell + 1} ,$$

and

$$A_5 = \frac{2\zeta}{3} .$$

The amplification from the epoch a_i $(a_\mathrm{i} \ll a_\mathrm{e})$ to the radiation era is thus

$$\alpha = \frac{1}{3}(2\ell + 1)\left(\frac{a_\mathrm{e}}{a_\mathrm{i}}\right)^{2\ell} . \qquad (50)$$

The explicit dependence in (50) of the amplification on the initial epoch a_i shows that amplification, and thus the spectrum, is not scale invariant.

The density perturbations are given using equations (35), (46) and (47). For modes beyond the Hubble radius during inflation

$$\frac{\delta\rho}{\rho} \approx -2(2\ell + 1)A_4\left(\frac{a}{a_\mathrm{e}}\right)^{2\ell} , \qquad (51)$$

while for super-horizon modes during the radiation dominated era $(a \gg a_\mathrm{e})$

$$\begin{aligned}
\frac{\delta\rho}{\rho} &\approx & 2i\sqrt{2\ell}\left(\frac{a_\mathrm{e}}{a}\right)^{\ell+3/2} Z_{(2\ell-3)/2\ell}\left[-i\sqrt{\frac{2}{\ell}}\left(\frac{a_\mathrm{e}}{a}\right)^{\ell}\right] \\
& & -2\left(\frac{a_\mathrm{e}}{a}\right)^{3/2} Z_{-3/2\ell}\left[-i\sqrt{\frac{2}{\ell}}\left(\frac{a_\mathrm{e}}{a}\right)^{\ell}\right] ,
\end{aligned} \qquad (52)$$

which leads to

$$\frac{\delta\rho}{\rho} \approx -2A_5 + 4B_5\left(\frac{a_\mathrm{e}}{a}\right)^3 . \qquad (53)$$

During inflation $(a \ll a_\mathrm{e})$, $\delta\rho/\rho$ grows as $a^{2\ell}$, while in the radiation dominated era $(a \gg a_\mathrm{e})$, $\delta\rho/\rho$ is approximately constant. The amplification is the same as (50).

The evolution of the tensor perturbations for super-horizon modes is given by equation (16):

$$\tilde{h} = A_6 + B_6 \int \frac{(a^{2\ell} + a_\mathrm{e}^{2\ell})^{1/\ell} a_\mathrm{e}}{a^4} da . \qquad (54)$$

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For the inflationary era $(a \ll a_{\mathrm{e}})$ equation (54) gives

$$\tilde{h} \approx A_6 - \frac{1}{3} B_6 \left(\frac{a_{\mathrm{e}}}{a} \right)^3 , \tag{55}$$

while for the radiation dominated era $(a \gg a_{\mathrm{e}})$

$$\tilde{h} \approx A_6 - B_6 \left(\frac{a_{\mathrm{e}}}{a} \right) . \tag{56}$$

Unlike the scalar the tensor perturbations do not grow during inflation on super-horizon scales.

IV Sub-horizon perturbations

For perturbations on scales inside the Hubble radius (where $k/(aH)$ is not negligible), equations (23), (24), (25), (34), (35) and (36) can not be solved exactly. We can however give analytic forms for the solutions in inflation and radiation domination using [14, 15], since the equations reduce to Bessel forms.

A Power-law-like model

Defining the comoving wave numbers

$$k_1 \equiv \frac{a_{\mathrm{e}}(1 - n)}{t_{\mathrm{e}}(1 + 1/m)^{(1-2n)/2}} ,$$

$$k_2 \equiv \frac{a_{\mathrm{e}}}{2t_{\mathrm{e}}(1 + m)^{(1-2n)/2}} ,$$

we find the following. During power-law-like inflation $(t \ll m t_{\mathrm{e}})$, the scalar and tensor perturbations evolve as

$$\tilde{\Phi} \approx \left(\frac{t_{\mathrm{e}}}{t} \right)^{(n+1)/2} Z_{(n+1)/2(1-n)} \left[\frac{k}{k_1} \left(\frac{t}{t_{\mathrm{e}}} \right)^{1-n} \right] , \tag{57}$$

$$\frac{\delta\rho}{\rho} \approx \left[\frac{-2}{3} \left(\frac{k}{k_1} \right)^2 (1 - n)^2 \left(\frac{t}{t_{\mathrm{e}}} \right)^{(2-2n)} - 2 \right] \left(\frac{t_{\mathrm{e}}}{t} \right)^{(n+1)/2} Z_{(n+1)/2(1-n)} \left[\frac{k}{k_1} \left(\frac{t}{t_{\mathrm{e}}} \right)^{(1-n)} \right]$$
$$+ \frac{2(1-n)}{n} \frac{k}{k_1} \left(\frac{t}{t_{\mathrm{e}}} \right)^{(1-3n)/2} Z_{(3-n)/2(1-n)} \left[\frac{k}{k_1} \left(\frac{t}{t_{\mathrm{e}}} \right)^{(1-n)} \right] , \tag{58}$$

$$\tilde{h} \approx \left(\frac{t}{t_{\mathrm{e}}} \right)^{(1-3n)/2} Z_{(3n-1)/2(1-n)} \left[\frac{k}{k_1} \left(\frac{t}{t_{\mathrm{e}}} \right)^{1-n} \right] . \tag{59}$$

During radiation domination $(t \gg m t_{\mathrm{e}})$, we find the following:

$$\tilde{\Phi} \approx \left(\frac{t_{\mathrm{e}}}{t} \right)^{3/4} Z_{3/2} \left[\frac{k}{k_2} \left(\frac{t}{t_{\mathrm{e}}} \right)^{1/2} \right]$$
$$= \left(\frac{t_{\mathrm{e}}}{t} \right)^{3/2} \left\{ A_7 \sin \left[\frac{k}{k_2} \left(\frac{t}{t_{\mathrm{e}}} \right)^{1/2} \right] - B_7 \cos \left[\frac{k}{k_2} \left(\frac{t}{t_{\mathrm{e}}} \right)^{1/2} \right] \right\}$$
$$- \frac{k}{k_2} \left(\frac{t_{\mathrm{e}}}{t} \right) \left\{ A_7 \cos \left[\frac{k}{k_2} \left(\frac{t}{t_{\mathrm{e}}} \right)^{1/2} \right] + B_7 \sin \left[\frac{k}{k_2} \left(\frac{t}{t_{\mathrm{e}}} \right)^{1/2} \right] \right\} , \tag{60}$$

$$\frac{\delta\rho}{\rho} \approx \left[\frac{-8}{3} \left(\frac{k}{k_2} \right)^2 \frac{t}{t_{\mathrm{e}}} - 2 \right] \left(\frac{t_{\mathrm{e}}}{t} \right)^{3/4} Z_{3/2} \left[\frac{k}{k_2} \left(\frac{t}{t_{\mathrm{e}}} \right)^{1/2} \right]$$
$$+ 2 \frac{k}{k_2} \left(\frac{t_{\mathrm{e}}}{t} \right)^{3/4} Z_{5/2} \left[\frac{k}{k_2} \left(\frac{t}{t_{\mathrm{e}}} \right)^{1/2} \right]$$

8

$$
\begin{aligned}
&= \left[\frac{-8}{3}\left(\frac{k}{k_2}\right)^2 \frac{t}{t_\mathrm{e}} - 2\right]\left(\frac{t_\mathrm{e}}{t}\right)^{3/2}\left\{A_7 \sin\left[\frac{k}{k_2}\left(\frac{t}{t_\mathrm{e}}\right)^{1/2}\right] - B_7 \cos\left[\frac{k}{k_2}\left(\frac{t}{t_\mathrm{e}}\right)^{1/2}\right]\right\} \\
&\quad + \left[\frac{8}{3}\left(\frac{k}{k_2}\right)^2 \frac{t}{t_\mathrm{e}} + 2\right]\frac{t_e}{t}\frac{k}{k_2}\left\{A_7 \cos\left[\frac{k}{k_2}\left(\frac{t}{t_\mathrm{e}}\right)^{1/2}\right] + B_7 \sin\left[\frac{k}{k_2}\left(\frac{t}{t_\mathrm{e}}\right)^{1/2}\right]\right\} \\
&\quad - \frac{k}{k_2}A_8\left(\frac{t_\mathrm{e}}{t}\right)^2\left\{3\sin\left[\frac{k}{k_2}\left(\frac{t}{t_e}\right)^{1/2}\right] - 3\frac{k}{k_2}\left(\frac{t}{t_e}\right)^{1/2}\cos\left[\frac{k}{k_2}\left(\frac{t}{t_e}\right)^{1/2}\right]\right. \\
&\quad \left. - \left(\frac{k}{k_2}\right)^2 \frac{t}{t_\mathrm{e}}\sin\left[\frac{k}{k_2}\left(\frac{t}{t_e}\right)^{1/2}\right]\right\} \\
&\quad - \frac{k}{k_2}B_8\left(\frac{t_\mathrm{e}}{t}\right)^2\left\{3\cos\left[\frac{k}{k_2}\left(\frac{t}{t_e}\right)^{1/2}\right] + 3\frac{k}{k_2}\left(\frac{t}{t_\mathrm{e}}\right)^{1/2}\sin\left[\frac{k}{k_2}\left(\frac{t}{t_\mathrm{e}}\right)^{1/2}\right]\right. \\
&\quad \left. - \left(\frac{k}{k_2}\right)^2 \frac{t}{t_\mathrm{e}}\cos\left[\frac{k}{k_2}\left(\frac{t}{t_\mathrm{e}}\right)^{1/2}\right]\right\}, \tag{61}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{h} &\approx \left(\frac{t_\mathrm{e}}{t}\right)^{1/4} Z_{1/2}\left[\frac{k}{k_2}\left(\frac{t}{t_\mathrm{e}}\right)^{1/2}\right] \\
&= \left(\frac{t_\mathrm{e}}{t}\right)^{1/2}\left\{A_9 \sin\left[\frac{k}{k_2}\left(\frac{t}{t_e}\right)^{1/2}\right] - B_9 \cos\left[\frac{k}{k_2}\left(\frac{t}{t_e}\right)^{1/2}\right]\right\}. \tag{62}
\end{aligned}
$$

As expected, density perturbations oscillate after re-entry, since radiation pressure balances gravitational infall. Note that while there are non-decaying oscillations in the density perturbations, the tensor perturbation oscillations are purely decaying. At late times, the density perturbation (61) has the form

$$
\frac{\delta\rho}{\rho} \approx A\cos\left(\frac{k}{k_2}\frac{a}{a_\mathrm{e}}\right) + B\sin\left(\frac{k}{k_2}\frac{a}{a_\mathrm{e}}\right),
$$

in agreement with the standard results for the radiation era [16] (p. 152).

B   de Sitter-like model

Sub-horizon perturbations during inflation ($a \ll a_\mathrm{e}$) have the form

$$
\tilde{\Phi} \approx \left(\frac{a}{a_\mathrm{e}}\right)^{(2\ell-1)/2} Z_{-(2\ell+1)/2}\left(-\frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_\mathrm{e}}\frac{a_\mathrm{e}}{a}\right), \tag{63}
$$

$$
\begin{aligned}
\frac{\delta\rho}{\rho} &\approx -\frac{2}{3}\left(\frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_\mathrm{e}}\right)^2\left(\frac{a}{a_\mathrm{e}}\right)^{(2\ell-5)/2} Z_{-(2\ell+1)/2}\left(-\frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_\mathrm{e}}\frac{a_\mathrm{e}}{a}\right) \\
&\quad - \frac{2\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_\mathrm{e}}\left(\frac{a}{a_\mathrm{e}}\right)^{(2\ell-3)/2} Z_{-(2\ell+3)/2}\left(-\frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_\mathrm{e}}\frac{a_\mathrm{e}}{a}\right), \tag{64}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{h} &\approx \left(\frac{a_\mathrm{e}}{a}\right)^{3/2} Z_{-3/2}\left(-\frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_\mathrm{e}}\frac{a_\mathrm{e}}{a}\right) \\
&= A_{10}\left[\cos\left(\frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_\mathrm{e}}\frac{a_\mathrm{e}}{a}\right) + \frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_\mathrm{e}}\frac{a_\mathrm{e}}{a}\sin\left(\frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_\mathrm{e}}\frac{a_\mathrm{e}}{a}\right)\right] \\
&\quad - B_{10}\left[\sin\left(\frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_\mathrm{e}}\frac{a_\mathrm{e}}{a}\right) + \frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_\mathrm{e}}\frac{a_\mathrm{e}}{a}\sin\left(\frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_\mathrm{e}}\frac{a_\mathrm{e}}{a}\right)\right]. \tag{65}
\end{aligned}
$$

Sub-horizon perturbations during radiation domination ($a \gg a_\mathrm{e}$) are given by:

$$
\tilde{\Phi} \approx \left(\frac{a_\mathrm{e}}{a}\right)^{3/2} Z_{3/2}\left(\frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_\mathrm{e}}\frac{a}{a_\mathrm{e}}\right)
$$

$$
\begin{aligned}
&= A_{11} \left(\frac{a_{\rm e}}{a}\right)^3 \left[\sin\left(\frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_{\rm e}}\frac{a}{a_{\rm e}}\right) - \frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_{\rm e}}\frac{a}{a_{\rm e}}\cos\left(\frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k}\frac{a}{a_{\rm e}}\right)\right] \\
&\quad - B_{11}\left(\frac{a_{\rm e}}{a}\right)^3 \left[\cos\left(\frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_{\rm e}}\frac{a}{a_{\rm e}}\right) + \frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_{\rm e}}\frac{a}{a_{\rm e}}\sin\left(\frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_{\rm e}}\frac{a}{a_{\rm e}}\right)\right] , \qquad (66)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\delta\rho}{\rho} &\approx \left[-\frac{2}{3}\left(\frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_{\rm e}}\right)^2\left(\frac{a}{a_{\rm e}}\right)^2 - 2\right]\left(\frac{a_{\rm e}}{a}\right)^{3/2} Z_{3/2}\left(\frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_{\rm e}}\frac{a}{a_{\rm e}}\right) \\
&\quad + \frac{2\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_{\rm e}}\left(\frac{a_{\rm e}}{a}\right)^{1/2} Z_{5/2}\left(\frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_{\rm e}}\frac{a}{a_{\rm e}}\right) \\
&= \left[\frac{-2}{3}\left(\frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_{\rm e}}\right)^2\left(\frac{a}{a_{\rm e}}\right)^2 - 2\right]\left(\frac{a_{\rm e}}{a}\right)^3 A_{11}\left[\sin\left(\frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_{\rm e}}\frac{a}{a_{\rm e}}\right)\right. \\
&\quad \left. - \frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_{\rm e}}\frac{a}{a_{\rm e}}\cos\left(\frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_{\rm e}}\frac{a}{a_{\rm e}}\right)\right] \\
&\quad + \left[\frac{2}{3}\left(\frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_{\rm e}}\right)^2\left(\frac{a}{a_{\rm e}}\right)^2 + 2\right]\left(\frac{a_{\rm e}}{a}\right)^3 B_{11}\left[\cos\left(\frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_{\rm e}}\frac{a}{a_{\rm e}}\right)\right. \\
&\quad \left. + \frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_{\rm e}}\frac{a}{a_{\rm e}}\sin\left(\frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_{\rm e}}\frac{a}{a_{\rm e}}\right)\right] \\
&\quad - A_{12}\left(\frac{a_{\rm e}}{a}\right)^3\left[\left(\frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_{\rm e}}\frac{a}{a_{\rm e}}\right)^2 \sin\left(\frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_{\rm e}}\frac{a}{a_{\rm e}}\right)\right. \\
&\quad - 3\sin\left(\frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_{\rm e}}\frac{a}{a_{\rm e}}\right) \\
&\quad \left. + 3\frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_{\rm e}}\frac{a}{a_{\rm e}}\cos\left(\frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_{\rm e}}\frac{a}{a_{\rm e}}\right)\right] \\
&\quad + B_{12}\left(\frac{a_{\rm e}}{a}\right)^3\left[\left(\frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_{\rm e}}\frac{a}{a_{\rm e}}\right)^2 \cos\left(\frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_{\rm e}}\frac{a}{a_{\rm e}}\right)\right. \\
&\quad \left. - 3\cos\left(\frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_{\rm e}}\frac{a}{a_{\rm e}}\right) - \frac{3\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_{\rm e}}\frac{a}{a_{\rm e}}\sin\left(\frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_{\rm e}}\frac{a}{a_{\rm e}}\right)\right] , \qquad (67)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{h} &\approx \left(\frac{a_{\rm e}}{a}\right)^{1/2} Z_{1/2}\left(\frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_{\rm e}}\frac{a}{a_{\rm e}}\right) \\
&= \frac{a_{\rm e}}{a}\left[A_{13}\sin\left(\frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_{\rm e}}\frac{a}{a_{\rm e}}\right) - B_{13}\cos\left(\frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_{\rm e}}\frac{a}{a_{\rm e}}\right)\right] . \qquad (68)
\end{aligned}
$$

Once again, we see that the density perturbations contain non-decaying oscillations while the tensor perturbation oscillations are purely decaying. At late times, the density perturbation (67) has the form

$$
\frac{\delta\rho}{\rho} \approx A\cos\left(\frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_{\rm e}}\frac{a}{a_{\rm e}}\right) + B\sin\left(\frac{\pi}{2^{(1-\ell)/\ell}}\frac{k}{k_{\rm e}}\frac{a}{a_{\rm e}}\right) ,
$$

as in the power-law-like case.

V   Conclusions

Our main result is a comprehensive presentation of analytic solutions for the scalar and tensor perturbations on super- and sub-horizon scales for two classes of cosmology, where the scale factor evolves smoothly from an inflationary era to a radiation dominated era. These classes encompass parametrised power-law-like and de Sitter-like inflationary behaviour. Our results confirm the amplification of super-horizon scalar perturbations for all values of the parameters which define the classes of models. Explicit forms, given in equations (40) and (50), of the amplification for scalar perturbations show how the de

Sitter-like inflation is much more effective in amplification than the power-law-like inflation. On sub-horizon scales, the detailed form of oscillating modes after re-entry was found, showing that there are non-decaying modes for density perturbations, but not for tensor perturbations. Taken together, these results provide a unified catalogue of perturbation solutions in what can be taken as the two main types of inflationary models with smooth exit.

References

[1] R. Maartens and D. Tilley, Gen. Rel. Grav. **30**, 289 (1998).

[2] E. Gunzig, R. Maartens and A. Nesteruk, Class. Quantum Grav. **15**, 923 (1998).

[3] N. Deruelle and V. Mukhanov, Phys. Rev. D. **52**, 5549 (1995).

[4] V. Mukhanov, H. Feldman and R. Brandenberger, Phys. Rep. **215**, 203 (1992).

[5] R. Caldwell, Class. Quantum Grav. **13**, 2437 (1996).

[6] M. Götz, Mon. Not. R. Astron. Soc, to appear (1998).

[7] J. Martin and D. Schwarz, Phys. Rev. D. **57**, 3302 (1998).

[8] R. Maartens, D. Taylor and N. Roussos, Phys. Rev. D. **52**, 3358 (1995).

[9] H. Kodama and T. Hamazaki, Preprint, gr-qc/9712045, (1997).

[10] D. Lyth, Phys. Rev. D. **31**, 1792 (1985).

[11] L. Grishchuk, Phys. Rev. D. **50**, 7154 (1994).

[12] L. Grishchuk, Preprint, gr-qc/9801011 (1998).

[13] W. Unruh, Preprint, astro-ph/9802323 (1998).

[14] E. Kamke, *Differentialgleichen: LösungsMethoden Und Lösungen* (Stuttgart: Teubner) (1983).

[15] I. Gradshteyn and I, Ryzhik, *Tables of Integrals, Series, and Products* (London: Academic Press) (1980).

[16] T. Padmanabhan, *Structure formation in the Universe* (Cambridge: University press) (1993).